

SOME NEW CLASSES OF EQUIENERGETIC GRAPHS

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Abstract: The eigenvalue of a graph G is the eigenvalue of its adjacency matrix and the energy $E(G)$ of graph G is the sum of absolute values of its eigenvalues. Two non-isomorphic graphs G_1 and G_2 of the same order are said to be equienergetic if they have same energies. The complement of a graph G is the graph $\overline{G}$ with vertex set $V(G) = V(\overline{G})$ and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G . In the present work three pairs of equienergetic graphs have been obtained using graph complement.

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1. Introduction and Preliminaries

All the graphs considered here are simple, finite and undirected. For standard terminology and notations related to graph theory we follow Balakrishnan and Ranganathan [2] while for the concept related to algebra, we follow Lang [7]. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$.

The adjacency matrix $A(G)$ of a graph G with vertices $v_1, v_2, \dots, v_n$ is an $n \times n$ matrix $[a_{ij}]$ such that,

$$\begin{aligned} a_{ij} &= 1, \text{ if } v_i \text{ is adjacent with } v_j \\ &= 0, \text{ otherwise} \end{aligned}$$

The eigenvalues of adjacency matrix of graph G are known as eigenvalues of graph. The collection of eigenvalues of the graph with their multiplicities is known as spectrum of the graph. If $\lambda_1, \lambda_2, \dots, \lambda_k$ are the distinct eigenvalues of G with respective multiplicities $m_1, m_2, \dots, m_k$, then the spectrum of G is denoted by,

$$\text{spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}, \text{ where } \sum_{i=1}^k m_i = n$$

Two non-isomorphic graphs are said to be cospectral if they have the same spectra, otherwise they are known as non-cospectral. The energy $E(G)$ of a graph G is the sum of absolute values of the eigenvalues of graph G .

Hence,

$$E(G) = \sum_{i=1}^n |\lambda_i|$$

The concept of energy was introduced by Gutman [6]. A brief account of energy of graph can be found in Cvetković *et al.* [5] and Li *et al.* [8]. Two non-isomorphic graphs G_1 and G_2 of same order are said to be *equienergetic* if $E(G_1) = E(G_2)$. Obviously, co-spectral graphs are always equienergetic. Balakrishnan and Ranganathan [2] showed the existence of non-cospectral equienergetic graphs. In 2005 Stevanović [12] constructed equienergetic graphs of order $p \equiv 0(mod 5)$. A systematic computer aided study have been carried out for equienergetic trees by Brankov *et al.* [3] and Milijkoć *et al.* [9]. Vaidya *et al.* [13, 14] have obtained some new classes of equienergetic graph using various graph operations.

The complement of a graph G is the graph $\overline{G}$ with vertex set $V(G) = V(\overline{G})$ and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G . A graph G with $G \cong \overline{G}$ is called self-complementary graph. Recently, Ramane *et al.* [10] have obtained non self-complementary graphs for which $E(G) = E(\overline{G})$. Such graphs are known as *complementary equienergetic graphs*. By the computer aided search Akabar Ali *et al.* [1] investigated complementary equienergetic graphs of order at most 10.

Proposition 1.1. [11] *Let G be an r -regular graph of order n with the eigenvalues $r, \lambda_1, \lambda_2, \dots, \lambda_n$. Then the eigenvalues of $\overline{G}$ are $n - r - 1, -\lambda_2 - 1, \dots, -\lambda_n - 1$.*

Definition 1.2. The Cartesian product of graphs G and H is a graph, denoted as $G \times H$, whose vertex set is $V(G) \times V(H)$. Two vertices (u_1, v_1) and (u_2, v_2) in $G \times H$ are adjacent if $u_1 = u_2$ and v_1 and v_2 are adjacent in H or $v_1 = v_2$ and u_1 and u_2 are adjacent in G .

Definition 1.3. The Kronecker product of G and H is denoted by $G \otimes H$ with vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) in $G \otimes H$ are adjacent if and only if u_1 and u_2 are adjacent in G as well as v_1 and v_2 are adjacent in H .

Proposition 1.4. [2] If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of G and $\mu_1, \mu_2, \dots, \mu_m$ are the eigenvalues of H , then

i the eigenvalues of $G \times H$ are $\lambda_i + \mu_j$, $i = 1, 2, \dots, n; j = 1, 2, \dots, m$.

ii the eigenvalues of $G \otimes H$ are $\lambda_i \mu_j$, $i = 1, 2, \dots, n; j = 1, 2, \dots, m$.

Definition 1.5. The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G'' . Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G'' .

Definition 1.6. The extended shadow graph $D_2^*(G)$ of a connected graph G is constructed by taking two copies of G say G' and G'' . Join each vertex u' in G' to the neighbours of the corresponding vertex u'' and with u'' in G'' .

If G is a graph of order n then $D_2(G)$ and $D_2^*(G)$ are graphs of order $2n$ and, if G is an r -regular graph then $D_2(G)$ and $D_2^*(G)$ are also regular graphs with degrees $2r$ and $2r + 1$ respectively.

Proposition 1.7. [13] If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of G then $2n$ eigenvalues of $D_2(G)$ are $2\lambda_1, 2\lambda_2, \dots, 2\lambda_n$, 0 (n times).

Proposition 1.8. [14] If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of G then $2n$ eigenvalues of $D_2^*(G)$ are $2\lambda_1 + 1, 2\lambda_2 + 1, \dots, 2\lambda_n + 1, -1$ (n times).

Definition 1.9. Let G be a graph with the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The extended bipartite double graph, $Ebd(G)$ of a graph G is the bipartite graph with its partite sets $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ in which two vertices x_i and y_j are adjacent if $i = j$ or v_i and v_j are adjacent in G .

If G is a graph of order n then $Ebd(G)$ is of order $2n$ and, if G is an r -regular graph then $Ebd(G)$ is an $(r + 1)$ -regular graph.

Proposition 1.10. [4] Let G be a graph of order n with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then the eigenvalues of extended bipartite double graph $Ebd(G)$ are $\pm(\lambda_1 + 1), \pm(\lambda_2 + 1), \dots, \pm(\lambda_n + 1)$.

2. Main Results

Theorem 2.1. *If $G \not\cong K_n$ is such that regular graph then*

$$E(\overline{Ebd(G)}) = E(\overline{G \times K_2}) = E(\overline{\overline{G} \otimes K_2})$$

Proof. Let G be an r -regular graph with eigenvalues $r, \lambda_2, \lambda_2, \dots, \lambda_n$. Therefore, by Proposition 1.1 the eigenvalues of $\overline{G}$ are $n-r-1, -\lambda_2-1, \dots, -\lambda_n-1$. According to Proposition 1.10 and Proposition 1.4, the spectra of $Ebd(G)$, $G \times K_2$ and $\overline{G} \otimes K_2$ are respectively

$$\begin{aligned} Spec(Ebd(G)) &= \begin{pmatrix} r+1 & \lambda_2+1 & \cdots & \lambda_n+1 & -(r+1) & -(\lambda_2+1) & \cdots & -(\lambda_n+1) \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \\ Spec(G \times K_2) &= \begin{pmatrix} r+1 & \lambda_2+1 & \cdots & \lambda_n+1 & r-1 & \lambda_2-1 & \cdots & \lambda_n-1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \\ Spec(\overline{G} \otimes K_2) &= \begin{pmatrix} n-r-1 & -\lambda_2-1 & \cdots & -\lambda_n-1 & -n+r+1 & \lambda_2+1 & \cdots & \lambda_n+1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \end{aligned}$$

Again, by Proposition 1.1

$$\begin{aligned} Spec(\overline{Ebd(G)}) &= \begin{pmatrix} 2n-r-2 & -\lambda_2-2 & \cdots & -\lambda_n-2 & r & \lambda_2 & \cdots & \lambda_n \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \\ Spec(\overline{G \times K_2}) &= \begin{pmatrix} 2n-r-2 & -\lambda_2-2 & \cdots & -\lambda_n-2 & -r & -\lambda_2 & \cdots & -\lambda_n \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \\ Spec(\overline{\overline{G} \otimes K_2}) &= \begin{pmatrix} n+r & \lambda_2 & \cdots & \lambda_n & n-r-2 & -\lambda_2-2 & \cdots & -\lambda_n-2 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \end{aligned}$$

Thus,

$$E(\overline{Ebd(G)}) = E(\overline{G \times K_2}) = E(\overline{\overline{G} \otimes K_2}) = 2n-2 + \sum_{i=2}^n (|\lambda_i| + |\lambda_i + 2|)$$

Theorem 2.2. *If G is an r -regular graph then*

$$E(\overline{D_2^*(G)}) = E(\overline{G} \otimes K_2)$$

Proof. Let G be a regular graph with eigenvalues $r, \lambda_2, \lambda_2, \dots, \lambda_n$. Therefore, by Proposition 1.1 the eigenvalues of $\overline{G}$ are $n-r-1, -\lambda_2-1, -\lambda_3-1, \dots, -\lambda_n-1$.

According to Proposition 1.4 and Proposition 1.8, the spectra of $\overline{G} \otimes K_2$ and $D_2^*(G)$ are respectively,

$$\text{Spec}(\overline{G} \otimes K_2) = \begin{pmatrix} n-r-1 & -\lambda_2-1 & \cdots & -\lambda_n-1 & -(n-r-1) & \lambda_2+1 & \cdots & \lambda_n+1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

and

$$\text{Spec}(D_2^*(G)) = \begin{pmatrix} 2r+1 & 2\lambda_2+1 & \cdots & 2\lambda_n+1 & -1 \\ 1 & 1 & \cdots & 1 & n \end{pmatrix}$$

Therefore, by Proposition 1.1 the spectrum of $\overline{D_2^*(G)}$ is,

$$\text{Spec}(\overline{D_2^*(G)}) = \begin{pmatrix} 2n-2r-2 & -2\lambda_2-2 & \cdots & -2\lambda_n-2 & -2\lambda_n-2 & 0 \\ 1 & 1 & \cdots & 1 & 1 & n \end{pmatrix}$$

Hence,

$$E(\overline{D_2^*(G)}) = 2n - 2r - 2 + 2 \sum_{i=2}^n |\lambda_i + 1| = E(\overline{G} \otimes K_2)$$

Remark 2.3. In [08], it was proved that $E(D_2) = 2E(G)$. Thus, If G is self complementary euenergetic graph then $E(G) = E(\overline{G})$.

$$\Rightarrow r + \sum_{i=2}^n |\lambda_i| = n - r - 1 + \sum_{i=2}^n |\lambda_i + 1|$$

In this case, $E(\overline{D_2^*(G)}) = E(\overline{G} \otimes K_2) = 2(E(G)) = E(D_2(G))$

Theorem 2.4. If G is an r - regular graph with n vertices then

$$E(\overline{Ebd(\overline{G})}) = E(\overline{\overline{G} \times K_2})$$

Proof. Let G be a regular graph with eigenvalues $r, \lambda_2, \dots, \lambda_n$. Therefore, by Proposition 1.1 the eigenvalues of $\overline{G}$ are $n-r-1, -\lambda_2-1, \dots, -\lambda_n-1$. By Proposition 1.10 and Proposition 1.4, spectra of $Ebd(\overline{G})$ and $\overline{G} \times K_2$ are respectively,

$$\text{Spec}(Ebd(\overline{G})) = \begin{pmatrix} n-r & -\lambda_2 & \cdots & -\lambda_n & -(n-r) & \lambda_2 & \cdots & \lambda_n \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$\text{Spec}(\overline{G} \times K_2) = \begin{pmatrix} n-r & -\lambda_2 & \cdots & -\lambda_n & n-r-2 & -\lambda_2-2 & \cdots & -\lambda_n-2 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Thus, by Proposition 1.1,

$$Spec(\overline{Ebd(\overline{G})}) = \begin{pmatrix} n+r-1 & \lambda_2-1 & \cdots & \lambda_n-1 & n-r-1 & -\lambda_2-1 & \cdots & -\lambda_n-1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

and

$$Spec(\overline{\overline{G} \times K_2}) = \begin{pmatrix} n+r-1 & \lambda_2-1 & \cdots & \lambda_n-1 & -n+r+1 & \lambda_2+1 & \cdots & \lambda_n+1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Now,

$$E(\overline{\overline{G} \otimes K_2}) = 2n - 2 + \sum_{i=2}^n |\lambda_i - 1| + \sum_{i=2}^n |\lambda_i + 1| = E(\overline{Ebd(\overline{G})})$$

3. Conclusion

The concept of graph energy have drawn attention of many researchers due to its applications in chemistry. We have investigated some new families equienergetic graphs using graph complement.

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